Lifetime distributions in the methods of non-equilibrium statistical operator and superstatistics

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Abstract. A family of non-equilibrium statistical operators is introduced which differ by the system age distribution over which the quasi-equilibrium (relevant) distribution is averaged. To describe the nonequilibrium states of a system we introduce a new thermodynamic parameter - the lifetime of a system. Superstatistics, introduced in works of Beck and Cohen [Physica A **322**, (2003), 267] as fluctuating quantities of intensive thermodynamical parameters, are obtained from the statistical distribution of lifetime (random time to the system degeneracy) considered as a thermodynamical parameter. It is suggested to set the mixing distribution of the fluctuating parameter in the superstatistics theory in the form of the piecewise continuous functions. The distribution of lifetime in such systems has different form on the different stages of evolution of the system. The account of the past stages of the evolution of a system can have a substantial impact on the non-equilibrium behaviour of the system in a present time moment.

PACS. 05.40.-a 02.50.Ey 05.10.Gg

1 Introduction

One of the most promising ways of development of the description of the non-equilibrium phenomena is provided by the method of the non-equilibrium statistical operator (NSO) [1,2,3]. In [4] a new interpretation of the NSO method is given, where the operation of taking the invariant part [1] or the use of the auxiliary "weight func-

tion" (in the terminology of [5,6]) in NSO are treated as averaging of the quasi-equilibrium statistical operator over the distribution of the past lifetime span (age) of the system. In [5,6] it is noted, that multiple choice of the "weight functions" can be taken. In [2] a uniform distribution over an initial moment t_0 is considered, which after the change of integration order reduces to the exponentially distributed weight function $p_q(u) = \exp\{-\varepsilon u\}$

in (2). Such distribution is the limiting case of the lifetime distribution [7], that is the distribution of the first passage time of a given level. The term "lifetime" denoting the time of the first passage of a level was used in [7]. Encountered in the literature is also the term "non-equilibrium relaxation time", as well as some others.

The form of the function $p_q(u) = \exp\{-\varepsilon u\}$ in (2) is connected with the form of the source in the Liouville equation for *NSO*. In [8,9] the sources in the Liouville equation different from that introduced in the *NSO* method in [1,2] are considered. The approach of the present paper differs from the methods used in [8,9]. But the use of the distribution of the system lifetime in the present work can be compared with that in [8] enlarging the set of macroobservables; besides the common physical macroobservables this approach includes additional ones, namely the life span (lifetime).

In [10] other alternative derivations of *NSO* are performed, following the ideas proposed by McLennan [11], and a relation with an earlier proposal by I. Prigogine [12] is discussed. The source in the Liouville equation can in principle take different forms. The form of a source used in [1,2,3] is a specific case which can be obtained under the assumption of the weak coupling limit of the interaction of the system with its environment.

In [13] it is shown, what is the impact of changing the system lifetime distribution to the non-equilibrium properties of system with finite volume. In the present work we consider infinitely big systems with infinite average lifetimes as well.

In [2] a physical interpretation of the exponential distribution for $p_q(u)$ is given: the system freely evolves as an isolated system governed by the operator of Liouville. In addition, the system undergoes random transitions at which its phase point representing the system spreads from one phase trajectory to another one in a random fashion with an exponential probability under the influence of a "thermostat", thus average intervals between random jumps increase infinitely. This feature reflects in the parameter of the exponential distribution tending to zero after the thermodynamic limiting transition. Real physical systems have finite sizes. The exponential distribution describes completely random systems. The influence of the surrounding on a system can have organised character as well, for example, for the systems in the non-equilibrium steady-states with input and output flows. The ways of the interaction of the system with surrounding can be different, therefore various choices of the functions $p_q(u)$ are justified.

Nonextensive statistical mechanics [14,15] can be regarded as an embedding of the common statistical mechanics into a more general framework. Many complex systems exhibit a spatio-temporally inhomogeneous dynamics that can be effectively described by a superposition of several statistics on different time scales, termed "superstatistics" [16,17]. Nonequilibrium situations are described by a fluctuating parameter β, which can be, for example, the inverse temperature. The generalization of the Boltzmann factor $\exp\{-\beta E\}$ was introduced in the

following form:

$$B(E) = \int_0^\infty \mathrm{d}\beta' \, f(\beta') \exp\{-\beta' E\}. \qquad (1)$$

The type of superstatistics induced depends on the probability distribution $f(\beta)$ of the parameter β. The special case of these superstatistics, with the function f in the form of gamma-distribution, appears in the nonextensive statistical mechanics [16,17,14,15], describing a number of physical phenomena which are not satisfactory described by the Boltzmann-Gibbs statistics. In the present work the superstatistics like (1) (together with its generalization) is obtained starting from the distribution which contains a lifetime of a statistical system as a thermodynamic variable [18,19,20,21], Section 5. This distribution has been applied earlier to the description of aerosol behaviour [22], and neutrons in a nuclear reactor [23].

In several works, for example in [16,17], the distribution function $f(\beta)$ is introduced as some continuous function expressed through arbitrary analytical form of the distribution of a random variable β. But the definition of the continuous density of distribution assumes its piecewise continuous character when the density of distribution has finite number of breaks. Real nonequilibrium systems, as a rule, are spatially non-uniform. This behaviour can be mathematically described by the piecewise continuous functions, the examples of which are given in the present work.

2 Modifications to the nonequilibrium description

In [4] a new interpretation of the method of the NSO is given. Let us consider now, what consequences follow from such interpretation of NSO. Setting various distributions for past lifetime of the system $p_q(u)$, we obtain a family of NSO, where the exponential distribution in Zubarev NSO is a particular choice of the form of the (arbitrary) lifetime distribution $p_q(u)$. The class of NSO from this family can be related to the class of the lifetime (or age) distributions (taken, for example, from the stochastic theory of storage processes, the theory of queues etc) and to the relaxation properties of that class of physical systems which is investigated. The general expression for NSO with an arbitrary distribution $p_q(u)$ is

$$\ln \varrho(t) = \int_0^\infty p_q(u) \ln \varrho_q(t-u,-u) \, \mathrm{d}u = \qquad (2)$$

$$= \ln \varrho_q(t,0) - \int_0^\infty \left(\int p_q(u) \, \mathrm{d}u \right) \frac{\partial \ln \varrho(t-u,-u)}{\partial u} \, \mathrm{d}u,$$

$$\ln \varrho_q(t,0) = -\Phi(t) - \sum_n F_n(t) P_n;$$

$$\ln \varrho_q(t,t_1) = e^{\{-t_1 H/i\hbar\}} \ln \varrho_q(t,0) e^{\{t_1 H/i\hbar\}};$$

$$\Phi(t) = \ln \mathrm{Sp} \exp\{-\sum_n F_n(t) P_n\},$$

where H is Hamiltonian, $\ln \varrho(t)$ is the logarithm of the NSO, $\ln \varrho_q(t,0)$ is the logarithm of the quasi-equilibrium (or relevant) distribution; the first time argument indicates the time dependence of the values of the thermodynamic parameters F_m; the second time argument t_2 in $\varrho_q(t_1,t_2)$ denotes the time dependence through the Heizenberg representation for dynamical variables P_m on which

$\varrho_q(t,0)$ can depend [1,2,3,5,6], integration by parts in time is carried out at $\int p_q(y)\,dy_{|y=0} = -1; \int p_q(y)\,dy_{|y\to\infty} = 0$. If $p_q(y) = \varepsilon\exp\{-\varepsilon y\}; \varepsilon = 1/\langle\Gamma\rangle$, then the expression (2) passes in *NSO* from [1,2]. In [4] the auxiliary weight function [5,6] $p_q(u) = \varepsilon\exp\{-\varepsilon u\}$ was interpreted as the density of the probability distribution of the lifetime Γ of a system. There Γ is a random variables of the lifetime of a system from the moment t_0 of its birth till the current moment t; $\varepsilon^{-1} = \langle t - t_0\rangle; \langle t - t_0\rangle = \langle\Gamma\rangle; u = t - t_0$. This value represents the age of system. The operator of internal time describing the age of a system was also introduced in I.Prigogine's works (for example, see [12]). If the interval $t - t_0 = u$ is large enough (that is the details of an initial condition as dependence on the initial moment t_0 are nonsignificant and nonphysical [1,2]), it is possible to introduce the minimal lifetime $\Gamma_{min} = \Gamma_1$ and to integrate in (2) over the interval (Γ_1, ∞). It is possible to specify many concrete expressions for lifetime distribution of a system, each of which possesses its own advantages. Each of these expressions induces some form of a source in the Liouville equation for the nonequilibrium statistical operator. In the general case of an arbitrary function $p_q(u)$ the source is:

$$J = p_q(0)\ln\varrho_q(t,0) + \int_0^\infty \frac{\partial p_q(y)}{\partial y}\Big(\ln\varrho_q(t-y,-y)\Big)\,dy \tag{3}$$

(if the value $p_q(0)$ diverges, it is necessary to choose the lower limit of integration equal to some $\Gamma_{min} > 0$). Such approach corresponds to the dynamic Liouville equation in the form of Boltzmann-Bogoliubov-Prigogine [5,6,12], containing dissipative items. In [24] it was noted that the

role of the form of the source term in the Liouville equation in *NSO* method has never been investigated.

Let us notice, that in the case when the value $\partial\ln\varrho_q(t-y,-y)/\partial y$ (the operator of entropy production σ [1,2]) in the second term of the r.h.s. (2) does not depend on y and can be taken out from the integration on y, this second term takes on the form $\sigma\langle\Gamma\rangle$, and the expression (2) thus does not depend on the form of the function $p_q(y)$. It is the case, for example, if $\varrho_q(t) \sim \exp\{-\sigma t\}, \sigma = const$. In [25] such a distribution is obtained from the principle of maximum of entropy applied to the set of average values of fluxes.

As it is known (for example, [7]), the exponential distribution for lifetime

$$p_q(y) = \varepsilon\exp\{-\varepsilon y\}, \tag{4}$$

used in the works of Zubarev [1,2], is the limiting distribution for the lifetime, valid for large times. Thus, in [1,2] the thermodynamic results are obtained, which in this context are valid for all systems.

For the *NSO* (2) with the function (4) in the form suggested by Zubarev the value in the second item is

$$-\int p_q(y)\,dy = \exp\{-\varepsilon y\} = 1 - \varepsilon y + (\varepsilon y)^2/2 - \dots = \tag{5}$$

$$1 - y/\langle\Gamma\rangle + y^2/2\langle\Gamma\rangle^2 - \dots.$$

Evidently the average lifetime tends to infinity, $\langle\Gamma\rangle \to \infty$, and the correlation (5) tends to unity.

Besides the exponential density of probability (4), the Erlang distributions (special or general form), gamma distributions etc (see [26,27]), as well as the modifications

considering subsequent composed asymptotic of the decomposition [28] can be used as candidates for the density of lifetime distribution. Gamma distributions describe the systems whose evolution acquires some stages (number of these stages is given by the order of gamma distribution). Considering actual stages in non-equilibrium systems (chaotic, kinetic, hydrodynamic, diffusive and so forth [29]), it is possible to justify the use of gamma distributions of a kind

$$p_q(y) = \varepsilon(\varepsilon y)^{k-1} \exp\{-\varepsilon y\}/\Gamma(k) \qquad (6)$$

($\Gamma(k)$ is gamma function, passing at $k = 1$ to exponential one (4)), and to understand their importance in the description of non-equilibrium properties. The piecewise-continuous distributions corresponding to different stages of evolution of the system will be used below.

More accurate specifying the shape of the function $p_q(u)$ in comparison with limiting exponential function (4) allows to describe in more detail the real stages of evolution of a system. Every form of the lifetime distributions has certain physical sense. In [13] additional terms to NSO in Zubarev form for the gamma distribution (6) are obtained.

3 Systems with infinite lifetime

The amendments to NSO in the Zubarev form in the Section 2 and in [13] are obtained for the systems of finite sizes and lifetimes. We will show further, how the same effects, involving the influence of the past of system on its present non-equilibrium state have impact on the sys-

tems with infinite lifetime, for example, for the systems of infinite volume after thermodynamic limiting transition.

Amendments to the unity term in the equation (5) in [13] become vanishingly small when the size of the system and its average lifetime tend to infinity, as in the model distribution (4) used in Zubarev NSO. For the systems of finite size but still exponential distribution these terms result in nonzero amendments to the expression (5). Thus, these additional terms to NSO and hence to the kinetic equations, kinetic coefficients and other non-equilibrium characteristics of the system are in fact the impact of the finiteness of size and lifetime of the system. They do not result merely from the choice of particular form of the lifetime distribution in the system. In what follows we investigate whether there exist such distributions of the lifetime of system for which the additional contribution to NSO differs from Zubarev NSO even for infinitely large systems with infinitely large lifetime.

We will consider several examples of choosing the function $p_q(u)$ in (2). We limit ourselves to the piecewise-continuous distributions, from where the results different from [1,2] follow. There are numerous experimental evidences of such changes of the distribution of lifetime of the system $p_q(u)$ on the time scale of the life span of the system. In [30,31] the transition of the distribution of the first passage processes from Gaussian regime to the non-Lévy behaviour in a specific time moment is shown. Real systems possess finite sizes and finite lifetime which implies the influence of surrounding on them. The fact that these sources do not vanish in a limit of infinitely large

systems, is related to the openness of the system, hence to the influence of its surrounding.

3a) We shall set

$$p_q(u) = \begin{cases} \dfrac{ka^k}{(u+a)^{k+1}}, & u < c; \\ b\varepsilon\exp\{-\varepsilon u\}, & u \geq c. \end{cases} \tag{7}$$

From the condition of the normalization $\int_0^\infty p_q(u) = 1$ the normalization multiplier is $b = e^{\varepsilon c}\left(\dfrac{a}{a+c}\right)^k$. Average lifetime $\langle\Gamma\rangle$ for the distribution (7) is equal to

$$\langle\Gamma\rangle = \frac{a}{k-1} + \left(\frac{a}{a+c}\right)^k\left[\frac{1}{\varepsilon}(1+\varepsilon c) - \frac{(kc+a)}{(k-1)}\right]. \tag{8}$$

The value (8) $\langle\Gamma\rangle \to \infty$ at $\varepsilon \to 0$. From (7) we find

$$-\int p_q(u) = \begin{cases} \left(\dfrac{a}{u+a}\right)^k, & u < c; \\ b\exp\{-\varepsilon u\}, & u \geq c. \end{cases} \tag{9}$$

The source in the right part of Liouville equation for the distribution (7) in accordance with the expression (3) equals

$$J = \frac{k}{a}\ln\varrho_q(t,0) - \int_0^c \frac{k(k+1)a^k}{(u+a)^{k+2}}S\,du -$$
$$e^{c\varepsilon}\left(\frac{a}{a+c}\right)^k\int_c^\infty \varepsilon^2 e^{-\varepsilon u}S\,du,$$

where $S = \ln\varrho_q(t-u,-u)$. The distribution of *NSO* (2) in the case of (7) is

$$\ln\varrho(t) = \ln_{zub}\varrho(t) + \Delta;$$

$$\Delta = \int_0^c\left[\left(\frac{a}{a+u}\right)^k - e^{-\varepsilon u}\right]\sigma\,du +$$
$$\int_c^\infty\left[e^{\varepsilon c}\left(\frac{a}{a+c}\right)^k - 1\right]e^{-\varepsilon u}\sigma\,du,$$

where $\sigma = \partial\ln\varrho(t-u,-u)/\partial u$; $\ln_{zub}\varrho(t) = \ln\varrho_q(t,0) + \int_0^\infty e^{-\varepsilon u}\sigma\,du$ is the distribution obtained by Zubarev in [1,2], and Δ is a finite amendment to it.

3b) We will consider now the distribution of the following kind:

$$p_q(u) = \begin{cases} d, & u < c; \\ \varepsilon\exp\{-\varepsilon u\}, & u \geq c. \end{cases} \tag{10}$$

From the condition of the normalization we find $d = \dfrac{1}{c}(1 - e^{-\varepsilon c})$. The average lifetime is

$$\langle\Gamma\rangle = \frac{dc^2}{2} + \frac{1}{\varepsilon}e^{-\varepsilon c}(1 + \varepsilon c). \tag{11}$$

The average lifetime $\langle\Gamma\rangle \to \infty$ at $\varepsilon \to 0$. The source (3) in the Liouville equation in this case equals

$$J = d\ln\varrho_q(t,0) - \int_c^\infty \varepsilon^2 e^{-\varepsilon u}S\,du.$$

The amendment to the Zubarev form of *NSO* is

$$\Delta = -\int_0^c\left[e^{-\varepsilon u} + \frac{1}{c}\left(1 - e^{-\varepsilon c}\right)u\right]\sigma\,du.$$

We see in this case, that an additional memory term of the system stems from the finiteness of its size, and the limited memory effect is observed. It is possible to consider other examples of the functions $p_q(u)$ which result in the limited memory effects.

3c) For the exponential density of distribution but with different intensities in different time intervals

$$p_q(u) = \begin{cases} \varepsilon_1\exp\{-\varepsilon_1 u\}, & u < c; \\ b\varepsilon_2\exp\{-\varepsilon_2 u\}, & u \geq c, \end{cases} \tag{12}$$

from the condition of the normalization it follows that $b = e^{c(\varepsilon_2-\varepsilon_1)}$;

$$\langle\Gamma\rangle = \frac{1}{\varepsilon_1}\left[1 - e^{-\varepsilon_1 c}(1+\varepsilon_1 c)\right] + \frac{1}{\varepsilon_2}e^{-\varepsilon_1 c}(1+\varepsilon_2 c). \tag{13}$$

At $\varepsilon_2 \to 0$, $\langle\Gamma\rangle \to \infty$.

$$J = \varepsilon_1\ln\varrho_q(t,0) - \int_0^c \varepsilon_1^2 e^{-\varepsilon_1 u}S\,du -$$
$$e^{c(\varepsilon_2-\varepsilon_1)}\int_c^\infty \varepsilon_2^2 e^{-\varepsilon_2 u}S\,du;$$

$$\Delta = \int\limits_0^c \left[e^{-\varepsilon_1 u} - e^{-\varepsilon_2 u} \right] \sigma \, du + \int\limits_c^\infty \left[e^{c(\varepsilon_2 - \varepsilon_1)} - 1 \right] e^{-\varepsilon_2 u} \sigma \, du \,;$$

$$\Delta_{\varepsilon_2 \to 0} \to \int_0^\infty \left[e^{-\varepsilon_1 u} - 1 \right] \sigma \, du \,.$$

The natural question is now why do the examples of this section differ from the examples of Section 2. In the interpretation of [2] it is the random value t_0 in $u = t - t_0$ that fluctuates. In [2] the limiting transition is performed for the parameter $\varepsilon, \varepsilon \to 0$ in the exponential distribution $p_q(u) = \varepsilon \exp\{-\varepsilon u\}$ after passing to the thermodynamic limit. In the interpretation of [4] this corresponds to the fact that the mean lifetime of the system $\langle \Gamma \rangle = \langle t - t_0 \rangle = 1/\varepsilon \to \infty$. But average intervals between random jumps infinitely incease, exceeding the lifetime of the system. Therefore a source term in the Liouville equation tends to zero. If the change of the distribution $p_q(u)$ caused by the influence of the surrounding, occurs on the time interval of the life span of the system, as in examples 3a)-3c), its impact remains even if the mean lifetime tends to infinity.

4 Application of the distributions of Section 3 to the conductivity

On the example of conductivity we will investigate, what are consequences of the change of the type of functions $p_q(u)$ and $\varrho(t)$ as compared to the exponential law for $p_q(u)$, used in [3].

The determination of the conductivity coefficient by the *NSO* method is considered in [32,33,34,35]. In this section we will describe the transport of charges in the electric field, as linear reaction on a mechanical perturbation, that is we regard the electric conductivity in the linear approximation, following the results of [3] and, as in [3], we limit ourselves to the important special case – the reaction of the equilibrium system to the spatially homogeneous variable field

$$\boldsymbol{E}^0(t) = \int_{-\infty}^\infty \frac{d\omega}{2\pi} e^{-i\omega t} \tilde{\boldsymbol{E}}^0(\omega).$$

The Hamiltonian of perturbation is given by

$$H_t^1 = -\boldsymbol{P}\boldsymbol{E}^0(t),$$

where $\boldsymbol{P}$ is the operator corresponding to the vector of polarization of the system. In the coordinate representation this operator is written as

$$\boldsymbol{P} = \sum_i e_i \boldsymbol{r}_i,$$

where e_i is the charge of the particle, and $\boldsymbol{r}_i$ is its position vector. The operator of current is

$$\boldsymbol{J} = \dot{\boldsymbol{P}} = e \sum_j \dot{\boldsymbol{r}}_j = \frac{e}{m} \sum_j \boldsymbol{p}_j,$$

where $\boldsymbol{p}_j$ is particle momentum, m is mass of a particle. We choose a model in which the Coulomb interaction is taken into account as a self-consistent screening of the field, i.e. we take $\boldsymbol{E} = \boldsymbol{E}^0$. The most essential difference from [3] consists in the replacement of the Laplace transformation used in [3], that is

$$\langle A; B \rangle_{\omega + i\varepsilon} = \int_0^\infty dt \, e^{i(\omega + i\varepsilon)t} (A(t), B)), \ (\varepsilon > 0), \quad (14)$$

where $(A(t), B(t')) = \int_0^1 dx \mathrm{Tr}\{\Delta A(t) \Delta B(t' + i\beta\hbar x) \varrho_{eq}\}$ is the time correlation function [3], by the another integral transformation. So, for the example *3a)* with the distribution of the form (7), (9) the expression (14) is replaced

by

$$\langle A; B\rangle_{\omega;a,k} + e^{\varepsilon c}\left(\frac{a}{a+c}\right)^k \langle A; B\rangle_{\omega+i\varepsilon;(c,\infty)}, \qquad (15)$$

where

$$\langle A; B\rangle_{\omega;a,k} = \int_0^c dt\, e^{i\omega t}\left(\frac{a}{a+t}\right)^k (A(t), B);$$

$$\langle A; B\rangle_{\omega+i\varepsilon;(c,\infty)} = \int_c^\infty dt\, e^{i(\omega+i\varepsilon)t}(A(t), B). \qquad (16)$$

We consider an isotropic environment in which the tensor of conductivity is diagonal. In [3] the expression for the Laplace transform of the kind (14) for the specific resistance $\rho(\omega)$ is obtaned:

$$\rho(\omega) = \frac{1}{\sigma(\omega)} = \frac{3V}{\beta(\boldsymbol{J}, \boldsymbol{J})}[-i\omega + M]; \qquad (17)$$

$$M = \frac{\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega+i\varepsilon}}{(\boldsymbol{J}, \boldsymbol{J}) + \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega+i\varepsilon}}, \qquad (18)$$

where V is the volume of the system, β is inverse temperature, $\sigma(\omega)$ is the scalar coefficient of conductivity. In the examples considered below in expressions (17)-(18) it is the value M that changes. Performing the operations of [3], with replacement of expression (14) by (15), in place of correlation (18) we obtain a more complicated expression of the kind

$$M = \frac{\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega;a,k} + \frac{i\omega}{i(\omega+i\varepsilon)} e^{\varepsilon c}\left(\frac{a}{a+c}\right)^k \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega+i\varepsilon;(c,\infty)}}{K};$$

$$K = (\boldsymbol{J}(0), \boldsymbol{J}) - \frac{k}{a}\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega;a,k+1} -$$

$$\left(1 - \frac{i\omega}{i(\omega+i\varepsilon)}\right) e^{i\omega c}\left(\frac{a}{a+c}\right)^k (\boldsymbol{J}(c), \boldsymbol{J}) +$$

$$+ \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega;a,k} + \frac{i\omega}{i(\omega+i\varepsilon)} e^{\varepsilon c}\left(\frac{a}{a+c}\right)^k \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega+i\varepsilon;(c,\infty)}.$$

At $\varepsilon \to 0$ and $\langle \Gamma\rangle \to \infty$, $K \to (\boldsymbol{J}(0), \boldsymbol{J}) - \frac{k}{a}\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega;a,k+1} +$

$$+ \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega;a,k} + \left(\frac{a}{a+c}\right)^k \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega;(c,\infty)}.$$

For the distribution (10) in *3b)* the Laplace transform of the kind (14) is replaced by

$$-d\langle A; B\rangle_{\omega;c,t} + \langle A; B\rangle_{\omega+i\varepsilon;(c,\infty)},$$

where $\langle A; B\rangle_{\omega;c,t} = \int_0^c dt\, e^{i\omega t} t(A(t), B)$, the value d is given in (10), (11), $\langle A; B\rangle_{\omega+i\varepsilon;(c,\infty)}$ is given in (16). Instead of (18) in this case we will get the expression

$$M = \frac{M_1}{K_1},$$

$$M_1 = \frac{i\omega}{i(\omega+i\varepsilon)}\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega+i\varepsilon;(c,\infty)} - d\left[\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega;c,t} + \langle \boldsymbol{J}; \dot{\boldsymbol{J}}\rangle_{\omega;c,t=1}\right],$$

$$K_1 = \frac{i\omega}{i(\omega+i\varepsilon)}\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega+i\varepsilon;(c,\infty)} - d\left[\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega;c,t} + \right.$$

$$\left. \langle \boldsymbol{J}; \dot{\boldsymbol{J}}\rangle_{\omega;c,t=1}\right] + \left(dc + \frac{i\omega}{i(\omega+i\varepsilon)} e^{-\varepsilon c}\right) e^{i\omega c}(\boldsymbol{J}(c); \boldsymbol{J}).$$

If $\varepsilon \to 0, \langle \Gamma\rangle \to \infty, d \to 0$, and

$$M = \frac{\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega;(c,\infty)}}{e^{i\omega c}(\boldsymbol{J}(c), \boldsymbol{J}) + \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega;(c,\infty)}}.$$

This expression at small values of c is close to (18). For the case *3c)* with the distribution $p_q(u)$ of the kind (12) the Laplace transform (14) is substituted by the opeation

$$\langle A; B\rangle_{\omega+i\varepsilon;(0,c)} + e^{c(\varepsilon_2 - \varepsilon_1)}\langle A; B\rangle_{\omega+i\varepsilon_2;(c,\infty)},$$

where $\langle A; B\rangle_{\omega+i\varepsilon;(0,c)} = \int_0^c dt\, e^{i(\omega+i\varepsilon)t}(A(t), B)$, $\langle A; B\rangle_{\omega+i\varepsilon_2;(c,\infty)}$ is given in (16) and the value M from (18) is replaced by

$$M = \frac{\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega+i\varepsilon_1;(0,c)} + e^{(\varepsilon_2-\varepsilon_1)}\frac{i(\omega+i\varepsilon_1)}{i(\omega+i\varepsilon_2)}\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega+i\varepsilon_2;(c,\infty)}}{K_2},$$

$$K_2 = (\boldsymbol{J}(0), \boldsymbol{J}) - e^{i(\omega+i\varepsilon_1)c}(\boldsymbol{J}(c); \boldsymbol{J}) + \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega+i\varepsilon_1;(0,c)} +$$

$$e^{c(\varepsilon_2-\varepsilon_1)}\frac{i(\omega+i\varepsilon_1)}{i(\omega+i\varepsilon_2)}\left[e^{i(\omega+i\varepsilon_2)c}(\boldsymbol{J}(c), \boldsymbol{J}) + \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}}\rangle_{\omega+i\varepsilon_2;(c,\infty)}\right].$$

At $\langle \Gamma \rangle \to \infty$ and $\varepsilon_2 \to 0$ the value M changes unessentially, taking on the form

$$M = \frac{\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}} \rangle_{\omega+i\varepsilon_1;(0,c)} + e^{-c\varepsilon_1} \dfrac{i(\omega+i\varepsilon_1)}{i\omega} \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}} \rangle_{\omega;(c,\infty)}}{K_3},$$

$$K_3 = (\boldsymbol{J}(0), \boldsymbol{J}) - e^{i(\omega+i\varepsilon_1)c}(\boldsymbol{J}(c), \boldsymbol{J}) + \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}} \rangle_{\omega+i\varepsilon_1;(0,c)} +$$
$$e^{-c\varepsilon_1} \frac{i(\omega+i\varepsilon_1)}{i\omega} \left[e^{i\omega c}(\boldsymbol{J}(c), \boldsymbol{J}) + \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}} \rangle_{\omega;(c,\infty)} \right].$$

At small values of ε_1 this expression is close to (18). From (13) it is seen that $\lim_{\varepsilon_2 \to 0} \varepsilon_2 \langle \Gamma \rangle = e^{-\varepsilon_1 c}$.

Let us summarise explicit results for Coulomb systems. Such systems were investigated by the *NSO* method in [36,37,38,39,40]. We will follow [36]. We will derive the expressions for the conductivity of a completely ionised Coulomb plasmas in a constant electric field. For simplicity we limit ourselves to the case of the plasma consisting of two subsystems, electrons and positive ions. An isothermal limit is considered when the characteristic thermalization time for the charge carriers is much less than the relaxation time of their composite momentum. The formula for the isothermal conductivity (at the frequency $\omega = 0$) from [36] is written as follows:

$$\frac{1}{\sigma} = \frac{\beta}{3V} \left(\frac{m}{e^2 n} \right)^2 \lim_{\varepsilon \to +0} \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}} \rangle_{i\varepsilon}, \qquad (19)$$

where V is system volume, $\beta = 1/T$ is inverse temperature, $m = m_e$ and e are mass and charge of electron, $n = n_e$ is average electron density. For the correlation function from (19) in an expression is obtained [36]:

$$\lim_{\varepsilon \to +0} \langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}} \rangle_{i\varepsilon} =$$
$$-\frac{1}{\beta} \left(\frac{e}{m} \right)^2 \Sigma_{\boldsymbol{k}} \boldsymbol{k}^2 v(\boldsymbol{k}) S_i(\boldsymbol{k}) \left[\lim_{\omega \to 0} \frac{1}{\omega} \text{Im} \frac{1}{\epsilon_e(\boldsymbol{k}, \omega)} \right],$$

where $\boldsymbol{k}$ is wave vector, $v(\boldsymbol{k}) = 4\pi e^2/\boldsymbol{k}^2$, $S_i(\boldsymbol{k})$, the equilibrium statistical structure factor of ions, $\epsilon_e(\boldsymbol{k}, \omega)$, equilibrium dielectric constant of an electronic subsystem.

If we pass from the correlation function $\langle \dot{\boldsymbol{J}}; \dot{\boldsymbol{J}} \rangle_{i\varepsilon}$ of a kind (14) to the correlation function of a kind (15) in a case *3a)* (7) at $\langle \Gamma \rangle \to \infty, \varepsilon \to 0$ in (8) we see that at $\omega = 0$, $\frac{1}{\sigma} = \frac{1}{\sigma_{zub}} \frac{a^k}{(a+c)^k}$, where $\frac{1}{\sigma_{zub}}$ is the expression for the conductivity (19), derived in [36]. For a case *3b)* with the distribution $p_q(u)$ (10) at $\langle \Gamma \rangle \to \infty, \varepsilon \to 0$, $\frac{1}{\sigma} = \frac{1}{\sigma_{zub}}$. For a case *3c)* with the distribution (12) at $\langle \Gamma \rangle \to \infty, \varepsilon_2 \to 0$ in (13), $\frac{1}{\sigma} = \frac{1}{\sigma_{zub}} e^{-\varepsilon_1 c}$.

Direct comparison of the obtained results to the experiment presents certain difficulties, since the parameters c, a, k, ε_1 are apriori unknown.

5 Superstatistics from distribution containing lifetime

In [41] it is pointed out that a nonequilibrium distribution is characterized by an additional parameter related to the deviation of a system from the equilibrium (caused by the field of gravity, electric field for dielectrics etc). In the present work we consider open nonequilibrium, stationary systems, certain point of metastable states. Investigations on spin glasses and other aging systems, where a "waiting time" plays an important role, allows to anticipate the usability of this approach with respect to them as well. In the present paper we suggest a new choice of an additional parameter in the form of the lifetime of a physical system which is defined as a first-passage time till the random

process $y(t)$ describing the behaviour of the macroscopic parameter of a system (energy, for example) reaches its zero value. The lifetime Γ_x (or Γ) is thus a random process which is subordinate (in terms of the definitions of the theory of random processes [42]) with respect to the master process $y(t)$,

$$\Gamma_x = inf\{t : y(t) = 0\}, \quad y(0) = x > 0\,.$$

This definition of the lifetime is taken from the apparatus of the theory of random processes where it is widely used in the theory of queues, stochastic theory of storage [43], Kramers problem of the escape rate out of a potential well [44,45] and so on. These questions are discussed in textbooks by van Kampen [46], Gardiner [47] and many other [48]. The lifetime plays part in the theory of phase transitions, chemical reactions, in the dynamics of complex biomolecules etc.

Using a maximum-entropy principle [49], it is possible to derive the form of the expression for microscopic (but coarse-grained) probability density in the extended phase space [18, 19, 20, 21]

$$\rho(z; E, \Gamma) = \exp\{-\beta E - \gamma \Gamma\}/Z(\beta, \gamma)\,, \qquad (20)$$

where

$$Z(\beta, \gamma) = \int \exp\{-\beta E - \gamma \Gamma\}\, \mathrm{d}z = \qquad (21)$$
$$\int\int \mathrm{d}E\, \mathrm{d}\Gamma\, \omega(E, \Gamma) \exp\{-\beta E - \gamma \Gamma\}$$

is the partition function, β and γ are Lagrange multipliers satisfying the equations for the averages

$$\langle E \rangle = -\frac{\partial \ln Z}{\partial \beta}\Big|_{\gamma}\; ; \quad \langle \Gamma \rangle = -\frac{\partial \ln Z}{\partial \gamma}\Big|_{\beta}\,. \qquad (22)$$

The distribution (20) with the lifetime contains two different time scales: the first relates to the energy E, and the second - to the lifetime itself Γ, this latter one accounts for large-scale time correlations and large-time changes in E by means of a thermodynamic conjugate to the lifetime value γ. The similar operation can be derived starting from NSO. The structure factor $\omega(E)$ is thus replaced by $\omega(E, \Gamma)$ - the volume of the hyperspace containing given values of E and Γ. The number of phase points between $\{E, E + \mathrm{d}E; \Gamma, \Gamma + \mathrm{d}\Gamma\}$ equals $\omega(E, \Gamma)\mathrm{d}E\, \mathrm{d}\Gamma$. The value thermodynamically conjugated to the lifetime is related to the entropy fluxes and entropy production which characterize the peculiarities of the nonequilibrium processes in an open thermodynamic system. If $\gamma = 0$ and $\beta = \beta_0 = 1/(k_B T_{eq})$, where k_B is the Boltzmann constant, T_{eq} is the equilibrium temperature, then the expressions (20)-(22) yield the equilibrium Gibbs distribution. One can thus consider (20)-(22) as a generalization of the Gibbs statistics towards the nonequilibrium situation. Such physical phenomena as the metastability, phase transitions, stationary nonequilibrium states are known to violate the equiprobability of the phase space points. The value γ can be regarded as a measure of the deviation from the equiprobability hypothesis. In general one might choose the value Γ as a subprocess of some other kind as chosen above. Mathematically the introduction of the lifetime means acquiring additional information regarding an underlying stochastic process; namely, exploring the (stationary) properties of its slave process beyond merely knowledge of its stationary distribution.

In the distribution (20) containing lifetime as a thermodynamic parameter, the probability for E and Γ is equal to

$$P(E, \Gamma) = \frac{e^{-\beta E - \gamma \Gamma} \omega(E, \Gamma)}{Z(\beta, \gamma)}. \qquad (23)$$

Having integrated (23) on Γ, we obtain the distribution of a kind

$$P(E) = \int P(E, \Gamma) d\Gamma = \frac{e^{-\beta E}}{Z(\beta, \gamma)} \int_o^\infty \omega(E, \Gamma) e^{-\gamma \Gamma} d\Gamma. \qquad (24)$$

The structural factor $\omega(E, \Gamma)$ has a sense of the joint probability for E and Γ, considered as a stationary distribution of this process. We shall write down

$$\omega(E, \Gamma) = \omega(E)\omega_1(E, \Gamma) = \omega(E) \sum_{k=1}^n R_k f_k(\Gamma, E). \qquad (25)$$

In the last equality (25) it is supposed, that there exist n classes of ergodic states in a system; R_k is the probability that the system is in the k-th class of ergodic states, $f_k(\Gamma, E)$ is the density of lifetime distribution Γ in this class of ergodic states (generally f_k depends on E). As a physical example for such situation (typical for metals or glasses) one can mention the potential of many complex systems. This case is considered in [50].

In [20,21] various models of superstatistics are obtained from (20)-(25). For example:

$$P(E) \sim \frac{e^{-\beta E}}{\left[1 + (q-1)\left(1 - e^{-\beta_0 r_0 E}\right)\right]^{\frac{1}{q-1}}};$$

$$P(E) \sim$$
$$\frac{e^{-\beta E}}{\left\{ 1 + (q-1)(1 - \left[1 + (q_1 - 1)\left(1 - e^{-\beta_0 r_0 E}\right)\right]^{\frac{-1}{q_1-1}})\right\}^{\frac{1}{q-1}}},$$

and so on.

We note that there is a similarity between the method of superstatistics, where the averaging is performed over a parameter β (for example, the inverse temperature), as in (1), and the method of NSO, where averaging is performed over the extension of past time $u = t - t_0$, as in (2) [51]. Expressions (1), (2) and (27) are described by the subordinated random processes [42]. The Zubarev approach claims that the source term should be infinitesimally small. The question is now whether a vanishing source term would yield results different from the superstatisrtics as well. In [16,17,52] a very simple example of the Brownian particle is considered. Its velocity v satisfies the linear Langevin equation $\dot{v} = -\gamma v + \sigma L(t)$ where $L(t)$ is Gaussian white noise, $\gamma > 0$ is friction constant, and the strength of the noise is controlled in a usual fashion by the parameter σ. The stationary probability density of v is Gaussian with average 0 and variance β^{-1}, where the parameter $\beta = \gamma/\sigma^2$ can be identified with the inverse temperature of the statistical mechanics (we assume that the Brownian particle has a unit mass). This simple situation completely changes if the parameters γ and σ in the stochastic differential equation are assumed to fluctuate as well. To be specific, let us adopt that either γ or σ or both fluctuate in such a way that $\beta = \gamma/\sigma^2$ is χ^2- distributed with degree n. This implies that the probability density of β is given by

$$f(\beta) = \frac{1}{\Gamma(n/2)} \left(\frac{n}{2\beta_0}\right)^{\frac{n}{2}} \beta^{\frac{n}{2} - 1} \exp\left(-\frac{n\beta}{2\beta_0}\right);$$

$$\beta_0 = \int_0^\infty \beta f(\beta) d\beta. \qquad (26)$$

In the Zubarev approach if the parameter $\varepsilon \to 0$ in the exponential distribution (4), the source in the Liouville equation vanishes. Relating to the distribution (26) it corresponds to $\beta_0 \to \infty$ and $(n/\beta_0) \to 0$. As $u \leftrightarrow \beta$, $p_q(u) \leftrightarrow f(\beta)$, $\langle \Gamma \rangle \leftrightarrow \beta_0$ in (1), (2), (26), the case $\langle \Gamma \rangle \to \infty$ corresponds to that $\beta_0 \to \infty$. Apparently, this is the limiting case of $\sigma^2 \to 0$, when no stochastic element is present, but the system is subject to the dynamical force only, as in the Liouville equation without random source.

6 Piecewise continuous distributions for functions R, f from correlations (24)-(25), (1)

In this section the distributions of lifetime, having a different shape on the different temporal intervals of evolution of the system, are considered. Such behaviour is characteristic for many physical systems. It is stressed in [29] that non-equilibrium systems can have different stages of evolution. In [53] it is shown that the first passage time probability density distribution changes depending on the value of control parameter. Solutions of the Kramers equation, which are related to the first passage time probability density distribution also depend on the control parameter. Such transitions in real systems are widely encountered, the aging of materials is just one of known examples thereto.

Let us write the expressions (24)-(25) in the form

$$P(E) = \frac{e^{-\beta E} \omega(E)}{Z(\beta, \gamma)} \int_0^\infty R(x) f_1(x, E) dx \, ; \qquad (27)$$

$$f_1(x, E) = \int_o^\infty e^{-\gamma \Gamma} f(x, E, \Gamma) d\Gamma \, .$$

We note, that the correlation (27) includes Laplace transform to which a probabilistic sense can be ascribed according to [54]. For f and f_1 from (27) it is possible to use the models [18,19,20,21] which leads to superstatistics of the kind $\exp\{-yE\}$. The similar approach developed in [7] allows to obtain the correlation value $\Gamma_0(y) \sim a e^{kyE}$ for the model of phase synchronization, which under certain conditions reduces to the form $\exp\{-yE\}$. The parameters a and k depend on the problem.

Let us consider this problem for a simple case of the function $R(y)$ from (27), allowing to write an obvious form of probability density where the function $R(y)$ represents a combination of delta-function and homogeneous distribution:

$$R(y) = \begin{cases} p\delta(y - a), & y < c, \quad a < c, \quad p < 1 \, ; \\ (1-p)m^{-1}, & 0 < c < y < c + m \, ; \\ 0 & y \geq m \, . \end{cases}$$

Then

$$p(E) = \int_0^\infty R(y) \frac{dy}{Z_1(1 + \gamma a e^{kyE})} = \frac{1}{Z_1} \times$$

$$\left[\frac{p}{(1 + \gamma a e^{kyE})} + (1-p) \left(m + \frac{1}{kE} \ln \left| \frac{\frac{1}{\gamma a} + e^{kEc}}{\frac{1}{\gamma a} + e^{kE(c+m)}} \right| \right) \right]$$

Let us choose now the expressions of a kind

$$P(E) = \int_0^\infty R(y) \frac{1}{Z} e^{-\kappa y E} \, dy$$

(coinciding with (1)) with piecewise continuous distribution of the function $R(y)$. If we set the function $R(y)$ in

the form of the gamma-distribution with different values of the parameters α and r in different areas,

$$R(y) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda E}, & y < c; \\ b \dfrac{\lambda^r}{\Gamma(r)} y^{r-1} e^{-\lambda E}, & y \geq c, \end{cases}$$

then $b = \dfrac{\Gamma(r)\Gamma(\alpha, \lambda c)}{\Gamma(\alpha)\Gamma(r, \lambda c)}$, and

$$p(E) = \frac{1}{Z\Gamma(\alpha)} \left[\gamma(\alpha, (\kappa E + \lambda)c) \frac{1}{(1 + \kappa E/\lambda)^\alpha} + \quad (28) \right.$$
$$\left. \frac{\Gamma(\alpha, \lambda c)}{\Gamma(r, \lambda c)} \Gamma(r, (\kappa E + \lambda)c) \frac{1}{(1 + \kappa E/\lambda)^r} \right].$$

This distribution is more complex, than Tsallis distribution [14,15], obtained from gamma-distribution by means of a method of superstatistics [16,17]. Multipliers in a form $1/(1 + \kappa E/\lambda)^\alpha$ correspond to the Tsallis distribution, but in (28) other factors depending on E are present. The distribution (28) passes in $1/(1 + \kappa E/\lambda)^\alpha$ at $r \to \alpha$.

If

$$R(y) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda E}, & y < c; \\ b \dfrac{k a^k}{(y+a)^{k+1}}, & y \geq c, \end{cases}$$

then $b = \dfrac{k(a+c)^k \Gamma(\alpha, \lambda c)}{\Gamma(\alpha)}$, and

$$p(E) = \frac{1}{Z\Gamma(\alpha)} \left[\gamma(\alpha, (\kappa E + \lambda)c) \frac{1}{(1 + \kappa E/\lambda)^\alpha} + \right.$$
$$\left. k(a+c)^k \Gamma(\alpha, \lambda c) e^{a\kappa E} (\kappa E)^k \Gamma(-k, \kappa E(c+a)) \right].$$

Other similar examples can be considered as well. One could set three and more areas of piecewise change of variables (for example, two areas corresponding to different phases and a transitive layer between them). A continuous change of parameters can be treated on equal footing, for example, considering the parameter of the gamma-distribution continuously changing with some distribution

function or to be set by functions of a kind $R(g(\cdot))$. A combination of discrete R_k and continuous $R(y)$ distributions in different areas is also possible.

Let us consider two more simple examples of the combination of gamma-distribution with delta-distribution and with homogeneous distribution:

$$R(y) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda E}, & y < c; \\ b\,\delta(y-d), & y \geq c, d > c. \end{cases}$$

Then $b = \dfrac{\Gamma(\alpha, \lambda c)}{\Gamma(\alpha)}$,

$$p(E) = \frac{1}{Z\Gamma(\alpha)} \left[\gamma(\alpha, (\kappa E + \lambda)c) \frac{1}{(1 + \kappa E/\lambda)^\alpha} + \right.$$
$$\left. \Gamma(\alpha, \lambda c) e^{-d\kappa E} \right].$$

By

$$R(y) = \begin{cases} \dfrac{\lambda^\alpha}{\Gamma(\alpha)} y^{\alpha-1} e^{-\lambda E}, & y < c; \\ b m^{-1}, & 0 < c < y < c + m; \\ 0, & y \geq m, \end{cases}$$

$$b = \frac{\Gamma(\alpha, \lambda c)}{\Gamma(\alpha)},$$

$$p(E) = \frac{1}{Z\Gamma(\alpha)} \left[\gamma(\alpha, (\kappa E + \lambda)c) \frac{1}{(1 + \kappa E/\lambda)^\alpha} + \right.$$
$$\left. \Gamma(\alpha, \lambda c) e^{-c\kappa E} (\kappa E)^{-1} (1 - e^{-\kappa Em}) \right].$$

Various combinations of functions of distribution from [16,17] can be brought into consideration: for example, lognormal superstatistics, gamma superstatistics and inverse gamma superstatistics.

7 Applications

Passing to superstatistics we get different distributions $R(y)$ from (27) at $y < c$ and $y > c$ or $f(\beta)$ from (1) at $\beta < c$ and $\beta > c$. For example, exponential or gamma-distribution at $\beta < c$ and Pareto distribution corresponding to Tsallis distribution at $\beta > c$. A multitude of various combinations of different distributions can be encountered, including, for example, gamma-distribution with various parameters at different temperatures, the subordinated distributions and so on. The parameters of gamma-distribution can change discretely, but can be continuous as well.

In [55] it is assumed that Beck and Cohen's superstatistics provides a suitable description for systems with mixed regular-chaotic dynamics. Such systems can be described by means of the suggested approach. Examples of different behaviour of systems at different temperatures are obvious including superconductivity, and superfluidity, and other phase transitions.

The approach which is more general, than the superstatistics theory, consists in setting piecewise continuous distributions for $R(y)$ from expression (27), that is the probabilities for a system to be in in the k-th state.

Such distributions can describe laminar and turbulent modes in the stream, for example, the diffusion of the tobacco smoke flow in the atmosphere. For this case the index k (the parameter y - in a continuous case) corresponds to the spatial coordinate of the flow. In any point c there is a transition from a laminar mode to turbulent. The distribution $R(y)$ or $f(\beta)$ can be described by the correlations obtained in [56] (lognormal distribution). The situation described here is more general, than the only one transition at a certain temperature, as in the case of superstatistics.

In [57] the entropic index can take various values, as in the present work. In [58] the entropic parameter q of the Tsallis distribution depend on the parameter m, and the parameter m is used to give account of an energy loss rate or energy dissipation rate (or perhaps, the energy absorption rate). The Tsallis distribution appears to depend on the parameter m. Such situation corresponds to the approach of the present work.

The distribution suggested in [59] represents a special case of the piecewise continuous distribution used in the present work. In [60] the problems close to the problems of the present work are considered: the occupation of the accessible phase space (or of a symmetry-determined nonzero-measure part of it), which in turn appears to determine the entropic form to be used.

The results similar to results of Section 5 are obtained in [61]. For example, the distribution $f(\rho_1 \ldots \rho_N)$ of metastable states with local fluid densities ρ_i in different spatial domains $i = 1 \ldots N$ depending from the exponent λ is again related to the distance from the conventional equilibrium, as does the value γ from Section 5.

Besides the spatial heterogeneity the piecewise continuous distribution can describe the time changes. The suggested approach allows to use the methods of the theory of random processes for treating specific problems; for example, to refer to the stochastic theory of storage [43,

62], setting rates of an input in a potential well and an output from it, and to the methods of the Kramers escape problem [44,45].

8 Conclusion

As it is stated in [63], the existence of different time scales and the flow of the information from slow to fast degrees of freedom create the irreversibility of the macroscopical description. The information thus is not lost, but passes in the form inaccessible at the Markovian level of description. For example, for the rarefied gas the information is transferred from one-particle observable to multipartial correlations. In [4] the values $\varepsilon = 1/\langle \Gamma \rangle$ and $p_q(u) = \varepsilon \exp\{-\varepsilon u\}$ are expressed through the operator of entropy production and, according to the results of [63], in terms of the flow of the information from relevant to irrelevant degrees of freedom. The introduction of the function $p_q(u)$ in *NSO* corresponds to the specification of the description by means of the effective account of communication with irrelevant degrees of freedom. In the present work it is shown, how it is possible to expand the specification of the description of memory effects within the limits of *NSO* method. A more detailed description of the influence of fast varying variables on the evolution of system is suggested based on specifying the density of the life span distribution of a system.

In many physical problems the finiteness of the lifetime can be neglected. Then $\varepsilon \sim 1/\langle \Gamma \rangle \to 0$. For example, for a case of the evaporation of liquid drops it is possible to show [64], that non-equilibrium characteristics depend

on $\exp\{y^2\}$; $y = \varepsilon/(2\lambda_2)^{1/2}$, λ_2 is the second moment of the correlation function of fluxes averaged over quasi-equilibrium distribution. Estimations show, that even at the minimal values of lifetime of drops (generally of finite size) the maximum sizes are $y = \varepsilon/(2\lambda_2)^{1/2} \leq 10^{-5}$. Therefore finiteness of values of $\langle \Gamma \rangle$ and ε does not influence the behaviour of system and it is possible to consider $\varepsilon = 0$. However in some situations it is necessary to take into account the finiteness of lifetime $\langle \Gamma \rangle$ and values $\varepsilon > 0$. For example, this is the case of the nano-drops.

Changes of the form of the source in the Liouville equation, as well as the expressions for the kinetic coefficients, average fluxes, and kinetic equations can be obtained with the use of the *NSO*. It is possible to choose a class of lifetime distributions for which after thermodynamic limiting transition and tending the average lifetime of system to infinity the results are reduced to those obtained under exponential distribution for lifetime, used by Zubarev. However there is also another extensive class of realistic distributions of lifetime of system for which even if the average lifetime of system tends to infinity the non-equilibrium properties essentially change. It is a consequence of the interaction of the system with its environment.

For the distributions of the kind (7), (10), (12), having different form for different argument spans, non-vanishing corrections to Zubarev *NSO* persist even for infinitely large systems with infinitely large lifetimes. In the present work it is shown, that this situation is possible, for example, for the distributions of lifetime of a system, having different form at different stages of the evolution of a sys-

tem. Such behaviour corresponds to the realization of the evolution of a system in a number of subsequent stages [29].

It is interesting to investigate possibilities of such a choice of the function $p_q(u)$ which would in most full fashion correspond to physical conditions in which a system is placed. For an optimal choice of a form of function $p_q(u)$ it is possible to use the method of maximum entropy principle.

The use of the theory of the superstatistics in various applications is related to the piecewise continuous functions for the density of distribution of lifetime of the system. The task of these functions of distribution allows to receive new expressions for the superstatistics, corresponding to various physical situations. Similar results seem to be useful, for example, in the investigation of small systems. A number of results following from the interpretation of NSO and $p_q(u)$ as a density of lifetime distribution of system [4], can be obtained from the stochastic theory of storage [43] and theories of queues. For example, in [43] the general result that the random variable of the period of employment (lifetime) $g(u, x)$ has absolutely continuous distribution $p_q(u) \equiv g(u, x) = xk(u - x, u)$, $u > x > 0$, and $g(u, x) = 0$ otherwise, is stated. There $k(x, t)$ is an absolutely continuous distribution for the value $X(t)$ of the input into a system.

Pursuing the analogy between the methods of NSO and the superstatistics, it is necessary to consider in more detail a case of dynamical representation of the super-

statistics when $\beta_0 \to \infty$, and the distribution of an intensive parameter has a piecewise continuous character.

The form of distribution chosen by Zubarev for the life span represents a certain limiting case. The choice of the lifetime distribution in NSO is related to the account of the past of a system, its physical features, on the present moment; for example, with the account of the age of a system only, as in Zubarev form of NSO [1,2,3,4,5,6] at $\varepsilon > 0$, or with more detailed characteristic of the past evolution of a system. The obtained results are essential in cases when it is impossible to neglect the memory effects since the memory correlation time there is not vanishing. The analysis of the corresponding time scales is necessary as it is noted in [63].

The main objective of the present article is to show, how the systems with infinitely large average lifetime can induce nonvanishing sources in the Liouville equation, and in what consequences for the method of the NSO it results. Superstatistics with piecewise continuous distributions of intensive parameter are considered as well.

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